New approaches to Garland's method

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We may think of a simplicial complex as a geometric object:

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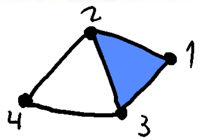

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for $\phi \in C_k(X)$ and $\sigma = \{v_0 < v_1 < \cdots < v_{k+1}\} \in X(k+1)$.

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Eckmann '44: $\tilde{H}_k(X; \mathbb{R}) \cong \ker L_k(X; w)$.
In particular, $\lambda_1(L_k(X; w)) > 0 \iff \tilde{H}_k(X; \mathbb{R}) = 0$.

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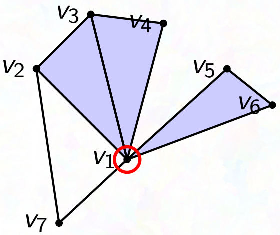

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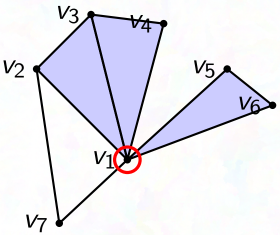

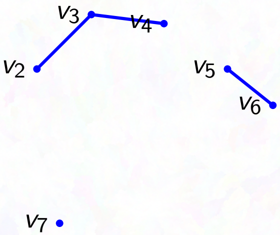

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Let $0 \leq \ell < k \leq d$. Then,

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In particular, if

$$\lambda_1(\tilde{L}_{k-\ell-1}(\mathrm{lk}(X,\sigma))) > \frac{(\ell+1)(d-k)}{k+1}$$

for all $\sigma \in X(\ell)$, then $\tilde{H}_k(X;\mathbb{R}) = 0$.

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for all $\sigma \in X(d-2)$, then $\tilde{H}_{d-1}(X;\mathbb{R}) = 0$. Note that $\mathrm{lk}(X,\sigma)$ is a graph in this case.

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- Extension of Garland's theorem to posets (Kauffman-Tessler '25, Babson-Welker '23+).

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- Write $\left\langle \tilde{L}_k(X)\phi, \phi \right\rangle$ in terms of $\left\langle \tilde{L}_{k-1}(\mathrm{lk}(X, v))\phi_v, \phi_v \right\rangle$, for $v \in V$ (plus correction term).

Extended Garland theorem via interlacing

- For $M_i \in \mathbb{R}^{n_i \times n_i}$ $(i = 1, \ldots, m)$, let $\bigoplus_{i=i}^{m} M_i$ be a block diagonal matrix with blocks $M_1, \ldots, M_m$.

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L '25+.

Let X be a pure d-dimensional simplicial complex, and let $0 \leq \ell < k \leq d$. Then, for all $1 \leq i \leq |X(k)|$,

$$(k-\ell)\lambda_i(\tilde{L}_k(X)) \geq (k+1)\lambda_i \left(\bigoplus_{\sigma \in X(\ell)} \tilde{L}_{k-\ell-1}(\mathrm{lk}(X, \sigma)) \right) - (\ell+1)(d-k).$$

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Corollary (L '25+)

$$\dim(\tilde{H}_k(X;\mathbb{R})) \leq \sum_{\sigma\in X(\ell)} t_\sigma,$$

where $t_\sigma = \left(\#\text{ of eigenvalues of } \tilde{L}_{k-\ell-1}(\mathrm{lk}(X,\sigma)) \leq \frac{(\ell+1)(d-k)}{k+1}\right)$.

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Let $m \geq n$. If $A \in \mathbb{R}^{m \times m}$ is symmetric, and $S \in \mathbb{R}^{m \times n}$ such that $S^T S = I$. Then, for all $1 \leq i \leq n$,

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- Write $\tilde{L}_k(X)$ as $S^T \left(\bigoplus_{\sigma \in X(\ell)} \tilde{L}_{k-\ell-1}(\mathrm{lk}(X,\sigma)) \right) S$ (up to "correction term"), apply interlacing theorem.

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- Motivated by ideas from Babson-Welker ('23+).

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The clique complex (or flag complex) $X(G)$ of graph $G = (V, E)$:
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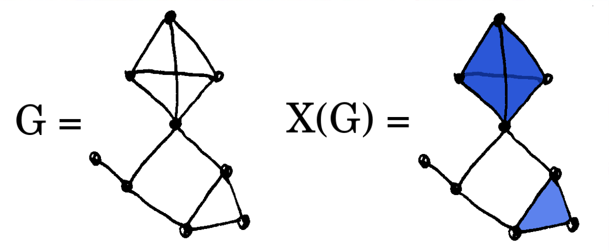
Example



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Aharoni-Berger-Meshulam '05:

Let $G = (V, E)$ be an n-vertex graph. Then, for all $k \geq 1$,

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As a consequence, if $\lambda_1(L_0(X(G))) = \lambda_2(L(G)) > \frac{kn}{k+1}$, then $\tilde{H}_k(X(G); \mathbb{R}) = 0$.

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Additive compound matrices (Wielandt '67)

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• Apply eigenvalue interlacing!

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• Find more applications of eigenvalue interlacing in the context of Garland-type results.

Thank you!