Sums of Laplacian eigenvalues and sums of degrees

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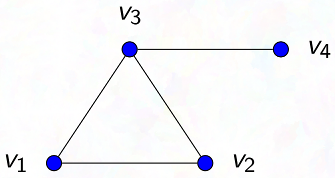

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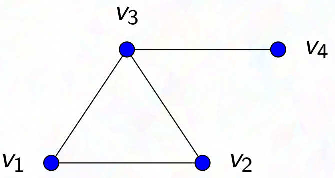

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

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- Laplacian eigenvalues of G provide upper bounds on dimension of homology groups of the clique complex of G (Aharoni, Berger, Meshulam '05, L '24).

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- Grone-Merris '94, Grone '95: $\sum_{i=1}^{k} \lambda_i(L(G)) \geq 1 + \sum_{i=1}^{k} d_i(G)$ (for $1 \leq k \leq n-1$).

Bai's theorem (Grone-Merris conjecture)

- Conjugate degree sequence of G: $d'_1(G) \geq d'_2(G) \geq \cdots \geq d'_n(G)$, where

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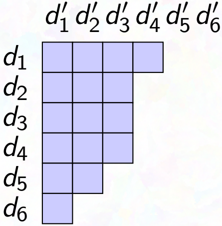

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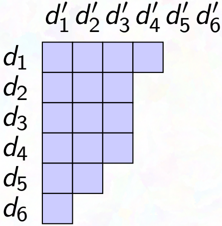

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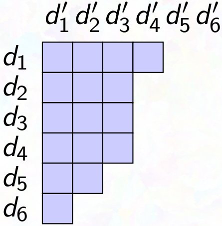

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Theorem (Bai '11). For $1 \leq k \leq n$,

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Theorem (L '24+)

$$\sum_{i=1}^{k} \lambda_i(L(G)) \leq |E| + \min\left\{2k^2 - \left\lceil \frac{k}{2} \right\rceil, k^2 + 15k\log k + 65k\right\}.$$

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Theorem (L '25+). For $1 \leq k \leq n/2$,

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– So $\sum_{i=1}^{k} \lambda_i(L(G)) = 2k = \sum_{i=1}^{2k} d_i(G)$.

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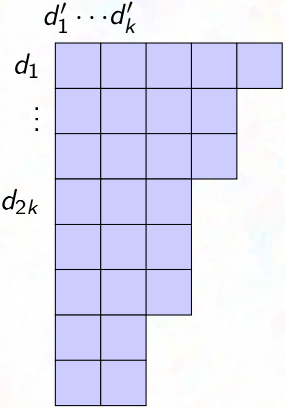

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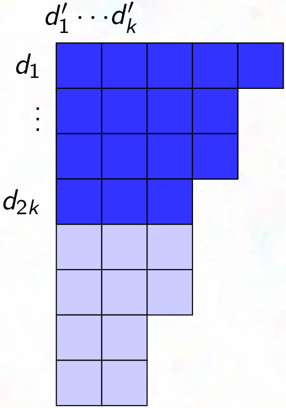

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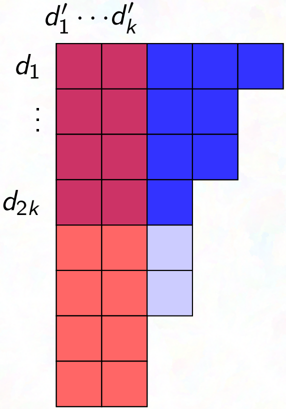

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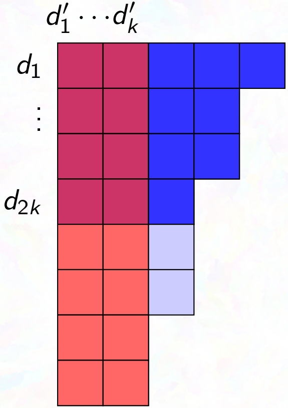

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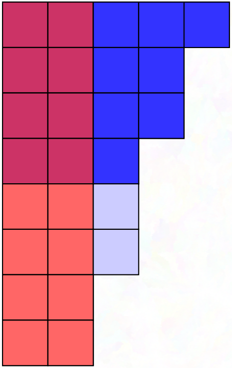
Proof

$d'_1 \cdots d'_k$

d_1

$\vdots$

d_{2k}



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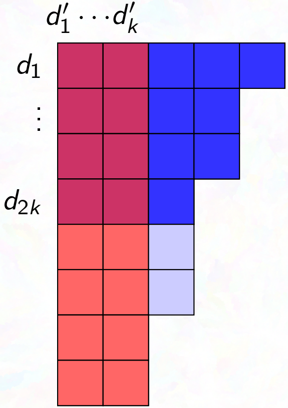

Recall Bai's theorem:
$\sum_{i=1}^{k} \lambda_i(L(G)) \le \sum_{i=1}^{k} d'_i(G)$.

So, averaging both bounds:

$$\begin{aligned}\sum_{i=1}^{k} \lambda_i(L(G)) &\le \frac{1}{2}\left(\sum_{i=1}^{2k} d_i(G) + \sum_{i=1}^{k} d'_i(G)\right)\\ &\le \frac{1}{2}(2|E| + 2k^2)\\ &= |E| + k^2\end{aligned}$$

Some applications

Proposition (L '25+)

For $1 \leq k \leq n/2$,

$$\sum_{i=1}^{k} \lambda_i(L(G)) \leq |E| + \max\{|E(G[S])| : \ S \subset V, \, |S| = 2k\}.$$

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In particular, the bound in Brouwer's conjecture holds for square-free graphs for $k \geq 7$, and for graphs with girth at least 5 for all $k \geq 1$.

Extensions and additional results

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• Using similar arguments, we prove a special case of an extension of the Grone-Merris conjecture to simplicial complexes due to Duval and Reiner.

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Theorem (Frobenius '12, Gershgorin '31). Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then,

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So,

$$\lambda_1(L^-(G)) \leq \max_{\{u,v\} \in E} \deg(u) + \deg(v) \leq d_1(G) + d_2(G).$$

General case – proof strategy

- Note: if $2k$ largest degrees of G are all equal to d, then

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- For a general graph, we decompose $L^-(G)$ into $2k$ matrices, one associated to each of the $2k-1$ vertices with highest degree, and an additional "leftover" matrix, for which the "trivial" argument above works!

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- Order the vertices as $v_1, \ldots, v_n$, where $\deg(v_i) = d_i(G)$.

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- Eigenvalues of L_i: $d_i(G), 0, \ldots, 0$.

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- The maximum "weighted degree" in L' is d, and by the Frobenius bound $\sum_{i=1}^{k} \lambda_i(L') \leq k\lambda_1(L') \leq 2kd$.

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By Fan's inequality,

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Some conjectures

Conjecture (L '25+)

Let $G = (V, E)$ be a graph, and let $1 \leq k \leq |V|$. Then,

$$\sum_{i=1}^{k} \lambda_i(L(G)) \leq |E| + \frac{k}{2} + \frac{1}{2}\sum_{i=1}^{k} \min\{d_i(G), k\}.$$

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- Note this would imply Brouwer's conjecture.

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• The following slightly weaker bound holds: $\sum_{i=1}^{k} \lambda_i(L(G)) \le |E| + k \cdot \nu(G) + \left\lfloor \frac{k}{2} \right\rfloor$, which is sharp if we don't enforce the "no isolated vertices" condition (L '24+).

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• If true, this would improve on the bound from Brouwer's conjecture for $k \geq \tau(G)$.

Thank you!